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Interaction of a free stream and a flow from a point source with dissimilar Bernoulli constants☆

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ABSTRACT

A method for solving problems when flows with dissimilar Bernoulli constants interact is proposed. The method is used to investegate the problem of a steady plane-parallel flow of an ideal incompressible fluid around a point source from which a fluid, with a density and overall pressure, differing from the corresponding free-stream values, enters. Calculations, carried out over the whole range of variation of the governing parameter, characterizing the energy of the fluid entering from the source, demonstrate the effectiveness of the method.

The basic difficulty in solving problems involving the interaction of flows with dissimilar Bernoulli constants is in finding the previously unknown interface line which is the line of tangential discontinuity of the velocity and is determined from the impermeability and pressure continuity conditions. When investigating of similar problems in the exact formulation, the traditional route is to introduce, for each fluid layer, its own domain of the parametric variable with subsequent derivation of the equations of the relation between the boundary points of these domains. This method has been used by a number of authors.[1–9]

A fundamentally different approach[10–12] consists of mapping the whole flow domain onto a certain canonical domain of the parametric plane. In this case, the required functions are chosen in such a way that the boundary conditions on segments which are not interface lines are satisfied by to the construction. This procedure turned out to be quite effective and has been successfully used.[13–16]

An additional serious mathematical difficulty, associated with the simulation of the flow close to the critical point, arises in solving problems involving the collision of fluid flows with dissimilar Bernoulli constants. A cuspidal point appears on the boundary of the flow domain (with the smaller Bernoulli constant), and it is difficult to use the method of conformal mappings in the neighbourhood of this point. Such problems have been investigated earlier.[6,8]

The problem of calculating the flow around a point source is, in fact, a model problem concerning the collision of jets. One of the procedures for solving this problem has been presented in a monograph,[6] where the solution is reduced to an iterative process for finding the two functions. In the numerical implementation of this process, simplifications are introduced into the formulae for the calculations close to the cuspidal point. Another method for solving the same problem has been proposed,[15] which is based on the realization of Sedov's idea of introducing a stagnation zone in the neighbourhood of the critical point, which enabled the above mentioned mathematical difficulties to be avoided by a modification of the flow model. This procedure led to the need to satisfy conditions for the closure of the boundary of the stagnation zone and to organize an additional iterative process. The problem of the projection of a symmetric wing profile with discharging a reactive flow towards a subsonic flow has also been solved by the introducing a stagnation zone in the neighbourhood of the critical point.[16]

A method for solving problems of the interaction of flows with dissimilar Bernoulli constants, which is illustrated by solving the problem of the flow around a source, is proposed below. The serious mathematical difficulties, associated with the simulation of the flow close to the critical cuspidal point, are successfully overcome by the choice of a new required function. The problem is solved without any of the simplifying assumptions or modifications of the flow model which have been used previously.[6,15] This method can be used to solve other direct and inverse problems concerned with the interaction and collision of flows. In a number of cases, it does not require the introduction of a parametric plane at all: the problem is directly solved in the flow domain. The number of required functions is minimal here and is the same as the number of boundary lines.

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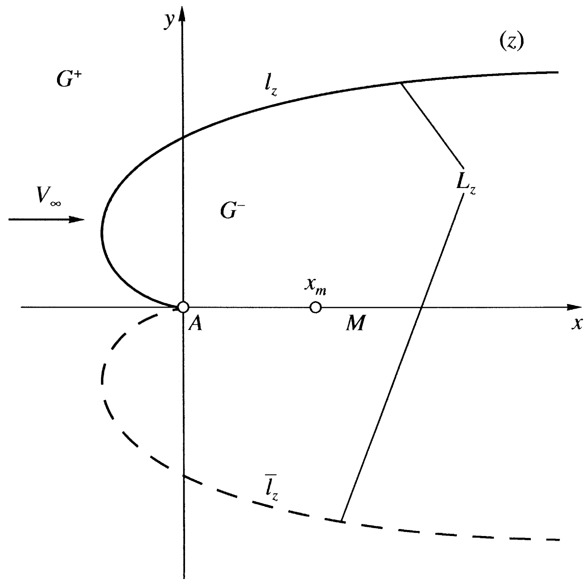

Fig. 1.

1. Formulation of the problem

In the physical plane $z = x + iy$ (Fig. 1), a steady plane-parallel flow of an ideal incompressible fluid c with density ρ and a velocity V_∞ at infinity flows past a point source M with a specified flow rate Q. The parameters of the fluid discharging from the source, the density ρ_J and the velocity $V_{\infty J}$ at infinity, differ from the corresponding free-stream and the subscript j is appended to them. The dimensionless parameter $\mu = \rho_j V_{\infty j}^2/(\rho V_\infty^2 - 1)$ characterizes the energy of the fluid discharging from the source. The interface line of the media L_z is the line of a tangential jump in the velocity, which is given by the formula

$$\rho_j V_j^2 = \rho V^2 + \mu\rho V_\infty^2 \tag{1.1}$$

that follows from the Bernoulli integrals for the two flows and the pressure continuity condition on passing through L_z. The origin of coordinates was chosen to be at the critical point A and the abscissa is directed along the free-stream velocity. The position of the source M will then be determined by the coordinate x_m which is not known in advance.

It is required to determine the shape of the interface line L_z of the media and the position of the critical point A with respect to the source, that is, to determine x_m.

2. Solution of the problem

We will denote the free-stream domain by G^+ and the jetflow domain by G^-. Under the assumptions made, a complex flow potential $w(z)$ exists in the domain G^+ and a potential $w_j(z)$ in the domain G^-.

Suppose $\zeta(\sigma)$ is a point on the line l_z (the upper half of the line L) with an arc abscissa measured from the point $A(0 \le \sigma \le \infty)$ and $\theta(\sigma)$ is the argument of the velocity vector at this point which, on the line l_z, is identical to the slope of the tangent (since l_z is a streamline). Then,

$$d\zeta = e^{i\theta(\sigma)}d\sigma \tag{2.1}$$

By virtue of condition (1.1) and the fact that the velocity of one or both the flows vanishes at the point A, the relation

$$\theta(0) = \theta_0 = \{0, \mu < 0; \pi/2, \mu = 0; \pi, \mu > 0\} \tag{2.2}$$

holds. At infinity, the line l_z must merge with the horizontal asymptote

$$y_\infty = Q/(2V_{\infty j}) \tag{2.3}$$

that is,

$$\theta(\infty) = 0 \tag{2.4}$$

Multiplying relation (1.1) by $e^{-2i/\theta}$, we obtain

$$\rho_j\left(\frac{dw_j}{dz}\right)^2 = \rho\left(\frac{dw}{dz}\right)^2 + \mu\rho V_\infty^2 e^{-2i\theta}$$

and, on then dividing by ρV_∞^2, we shall have

$$(1+\mu)\left(\frac{1}{V_{\infty j}}\frac{dw_j}{dz}\right)^2 = \left(\frac{1}{V_\infty}\frac{dw}{dz}\right)^2 + \mu e^{-2i\theta} \tag{2.5}$$

We now introduce the functions

$$\chi^{+}(z) = \left(\frac{1}{V_\infty}\frac{dw}{dz}\right)^2 - 1, \quad \chi^{-}(z) = (1+\mu)\left[\left(\frac{1}{V_{\infty j}}\frac{dw_j}{dz}\right)^2 - 1\right] \tag{2.6}$$

which are analytic in the domains G^+ and G^- respectively. The constants are chosen such that

$$\chi^{+}(\infty) = \chi^{-}(\infty) = 0 \tag{2.7}$$

Then,

$$\chi^{+}(\zeta) - \chi^{-}(\zeta) = 2\mu i \sin\theta e^{-i\theta} = \lambda(\zeta) \tag{2.8}$$

follows from equality (2.5).

If it is assumed that the function $\theta(\sigma)$ is known, then the line l_z can be found by integrating Eq. (2.1) and the function $\lambda(\zeta) = \lambda(\sigma)$ can be found from equality (2.8). Relation (2.8) is then a condition in the problem of determining the piecewise analytic function

$$\chi(z) = \{\chi^{+}(z), z \in G^{+}; \chi^{-}(z), z \in G^{-}\}$$

through the specified discontinuity (Refs. [17,31]) Taking account of the fact that the function $\chi^{-(z)}$ has a second order pole at the point $z = x_m$, we can write the solution of this problem in the form

$$\chi(z) = \Phi(z) + C_0 + \frac{C_1}{z - x_m} + \frac{C_2}{(z - x_m)^2}; \quad \Phi(z) = \frac{1}{2\pi i}\int_{L_z} \frac{\lambda(\zeta)d\zeta}{\zeta - z} \tag{2.9}$$

where $\Phi(z)$ is a Cauchy-type integral. Note that, by virtue of symmetry,

$$\Phi(z) = \Omega(z) + \overline{\Omega(\bar{z})}, \quad \Omega(z) = \frac{1}{2\pi i}\int_{l_z} \frac{\lambda(\zeta)d\zeta}{\zeta - z} \tag{2.10}$$

The representations (2.8)–(2.10) constitute the essence of the proposed method of solution and enable us to express the velocity fields in the flow domain in terms of a single unknown function $\theta(\sigma)$ which determines the shape of the interface line directly in the physical plane. Actually, the behaviour of the function $\chi(z)$ in the neighbourhood of the point M is known:

$$\chi(z)|_{z \to x_m} = (1+\mu)\left(\frac{Q}{2\pi V_{\infty j}(z - x_m)}\right)^2$$

whence it follows that

$$C_2 = \frac{(1+\mu)Q^2}{(2\pi V_{\infty j})^2}$$

In order to determine the constants C_0 and C_1, we find the first term of the expansion of the function $\chi(z)$ in powers of $1/z$ for large $|z|$. Using formulae (2.10), we obtain

$$\Phi(z) = \frac{1}{2\pi i z}\left[-\int_{l_z} \lambda(\zeta)d\zeta + \int_{l_z} \overline{\lambda(\zeta)d\zeta}\right] + o\left(\frac{1}{|z|}\right)$$

We now take account of relations (2.1) and (2.8) and derive the equality

$$\Phi(z) = -\frac{2\mu}{\pi z}\int_{l_z} \sin\theta(\sigma)d\sigma + o\left(\frac{1}{|z|}\right) = -\frac{2\mu y_\infty}{\pi z} + o\left(\frac{1}{|z|}\right)$$

and, when this and relation (2.7) are taken into account, we conclude from solution (2.9) that $C_0 = 0$. It now follows from equality (2.9) that the required expansion has the form

$$\chi(z) = \left(C_1 - \frac{2\mu y_\infty}{\pi}\right)\frac{1}{z} + o\left(\frac{1}{|z|}\right) \tag{2.11}$$

It follows from this and from the first formula of (2.6) that, in the domain G^+ for large $|z|$,

$$\frac{1}{V_\infty}\frac{dw}{dz} = 1 + \left(\frac{C_1}{2} - \frac{\mu y_\infty}{\pi}\right)\frac{1}{z} + o\left(\frac{1}{|z|}\right) \tag{2.12}$$

We now consider a semicircle of infinitely large radius in the upper half-plane and calculate the flow rate q of the fluid through the part of this half-plane belonging to the domain C_+. Using expansion (2.12), we find that

$$q = V_\infty\left[\left(\frac{C_1}{2} - \frac{\mu y_\infty}{\pi}\right)\pi - y_\infty\right]$$

However, the axis of symmetry and the curve l_z are streamlines and, therefore, $q = 0$.

Consequently, $C_1 = 2(1+\mu)y_\infty/\pi$ or, when account is taken of equality (2.3),

$$C_1 = \frac{(1+\mu)Q}{\pi V_{\infty j}} \tag{2.13}$$

whence

$$C_2 = \frac{C_1^2}{4(1+\mu)} \tag{2.14}$$

Note that, if the constant C_1 is specified in an arbitrary manner, then, in the case of the functions $(dw/dz)^2$ and $(dw)/(dz)^2$, first-order zeroes appear on the axis of symmetry. Their existence leads to the appearance of equipotential segments on this axis, which changes the whole flow pattern. Condition (2.13) is, in fact, a necessary condition for the absence of such zeroes.

According to formula (2.8) for λ and equalities (2.2), $\chi^+(0) = \chi^-(0)$ at the critical point. Then, recalling the definitions (2.6) and taking account of equality (2.2), we obtain

$$\chi^+(0) = \chi^-(0) = -a; \quad a = \begin{cases} 1+\mu, & \mu < 0 \\ 1, & \mu \geq 0 \end{cases} \tag{2.15}$$

Substituting expression (2.9) into this formula, we write the quadratic equation

$$[a + \Phi(0)]x_m^2 - C_1 x_m + C_2 = 0$$

from which, when account is taken of (2.14), we determine that

$$x_m = \frac{C_1}{2[a+\Phi(0)]}\left(1 \pm \sqrt{1 - \frac{a+\Phi(0)}{1+\mu}}\right) \tag{2.16}$$

Calculations showed that the radicand is always positive. It is necessary to take the root with the plus sign when $\mu < 0$ and the root with the minus sign when $\mu > 0$. If $\mu = 0$, then $a = 1$ and, according to definition (2.8), $\lambda(\zeta) \equiv 0$ and, definition (2.10), $\Phi(0) \equiv \Phi(z) = 0$. Consequently, the quadratic equation has one root ($x_m = Q/2\pi V_{\infty j}$)

Hence, if the function $\theta(\sigma)$ is found, the function $\chi(z)$ and, therefore, the complex conjugate velocities dw/dz and dw_j/dz are determined in the whole of the flow domain.

3. Iterative process scheme

The following iterative process is organized in order to find the unknown function $\theta(\sigma)$. The function $\theta(\sigma)$ must satisfy conditions (2.2) and (2.4). The initial approximation to it can be chosen in the form

$$\theta^{(0)}(\sigma) = \begin{cases} C\sigma^2\exp(-\sigma^2), & \mu < 0 \\ \theta_0[1 - 2\,\mathrm{arctg}(C\sigma)/\pi], & \mu \geq 0 \end{cases}$$

where C is a real positive constant and θ_0 is determined using formula (2.2).

The iterative procedure contains the following steps:

integrating Eq. (2.1), we find the line l_z, using formula (2.10), we determine $\Phi(\zeta)$, from equality (2.16), we find x_m, using formula (2.9) and taking account of the Sokhotskii formulae, we calculate $\chi^+(\zeta)$ and $\chi^-(\zeta)$ on the line l_z and, from the definitions (2.6) of the functions $\chi^+(z)$ and $\chi^-(z)$, we find a new approximation to the function $\theta(\sigma)$:

$$\theta(\sigma) = -\frac{1}{2}\arg(\chi_1(\sigma) + 1); \quad \chi_1(\zeta) = \begin{cases} \chi^+(\zeta), & \mu < 0 \\ \chi^-(\zeta), & \mu \geq 0 \end{cases}$$

It is necessary to take the branch of the function arg (z) from condition (2.4).

This iterative process has to be continued until the following condition is satisfied

$$\max_\sigma\left|\theta^{(n)}(\sigma) - \theta^{(n-1)}(\sigma)\right| < \varepsilon$$

where ε is a certain small positive number.

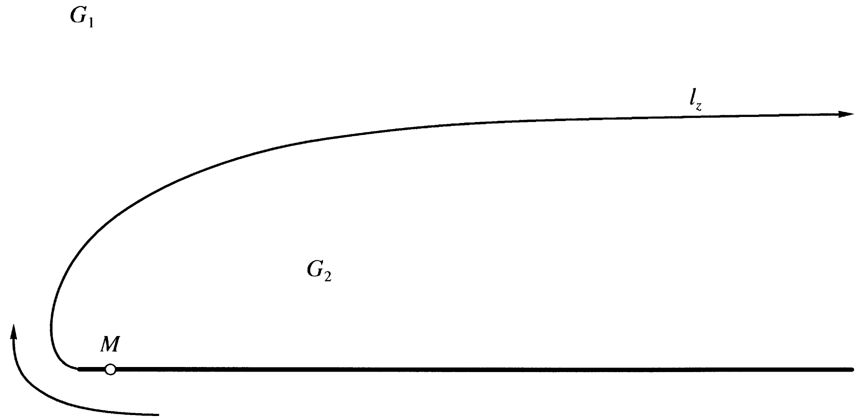

Fig. 2.

Note that this problem is characterized by just one dimensionless parameter μ, and the quantity $y_\infty = Q/(2V_{\infty j})$ can be taken as the characteristic length. At the same time, it can be assumed, without loss in generality, that $V_\infty = \rho = \rho_j = 1$ and, then,

$$\mu = V_{\infty j}^2 - 1 \tag{3.1}$$

4. Special cases

The solution of the problem in the special cases when $\mu = 0$ and $\mu = \infty$ can be successfully obtained analytically. If $\mu = 0$, then, according to equality (3.1), $V_{\infty j} = V_\infty = 1$, the Bernoulli constants of the two flows are the same, there is no tangential discontinuity in the velocity and there is a common complex potential, which is equal to the sum of the complex potentials of the external flow and the source.

In the case when $\mu = \infty$, the velocity V_j on the line l_z will be constant and equal to a certain value V_0, which can be arbitrarily assigned. The solution of this problem is easily constructed using jet theory methods. It can also be proved that the shape of the line l_z is identical in this special case to the shape of the free surface in the classical Helmholtz problem (historically, this is the first of the problems solved in the jet theory and it is a special case of the problem of flow in a Barda nozzle when the walls of the outer channel are removed to infinity (see Ref. [18]). If the Schiffman reflection method (Ref. [19], p.347) is applied to the lower half of a Helmholtz flow (the domain G_1, see Fig. 2), that is, the complex potential function is analytically extended across the free surface, then the upper half of the flow from the source (the domain G_2) is obtained. In this case, the line l_z will be a common stream line with a constant velocity.

5. Results of calculations

A series of calculations was carried out for different values of the parameter μ (see Fig. 3, for ease of comparison the origin of the coordinates was made coincident with the position of the source). The results of the calculations when $\mu \geq 0$ agreed with the results obtained earlier[6,15] and with the results in the special cases. Calculations were additionally carried out for the case when $-1 < \mu < 0$ which had not been previously investigated. The results are shown by the solid lines for $\mu > 0$, the dotted line for $\mu = 0$ and the dashed lines for $\mu < 0$. It has been noted [15] that, in the case when $\mu \geq 0$, the interface lines of the media have a common intersection point which is also

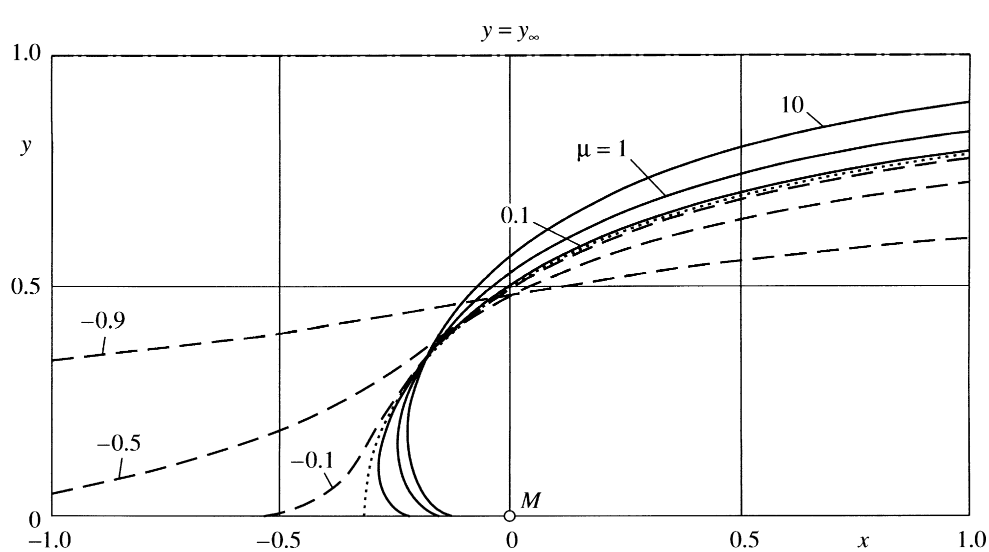

Fig. 3.

visible in Fig. 3. However, when $\mu < 0$, the lines l_z no longer pass through this point. We mention that, if $\mu \to -1$, then the flow in the domain G^+ degenerates into a homogeneous flow and $x_m \to \infty$.

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